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Higher order QED corrections to muon decay spectrum

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Abstract: QED radiative corrections to polarized muon decay spectrum are considered. Leading and next-to-leading logarithmic approximations are used. Exponentiation of soft radiation is discussed. The present theoretical uncertainty of the spectrum description is estimated.

Keywords: Standard Model, Weak Decays, Electromagnetic Processes and Properties.

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Contents

1. Introduction

Accurate measurements of the muon properties were providing substantial information for the development of the elementary particle physics during many years. Nowadays precision experiments with muons serve as one of the basements of the Standard Model (SM) and give a possibility to look for *new physics* [1, 2].

In this paper we discuss the present theoretical precision of the polarized muon decay spectrum description. The study is motivated by the experiment $\mathcal{TWIST}$ [3, 4], which is currently running at Canada's National Laboratory TRIUMF. The experiment is going to measure the spectrum with the accuracy level of about $1 \cdot 10^{-4}$. That will make a serious test of the space-time structure of the weak interaction. The experiment is able to put stringent limits on a bunch of parameters in models beyond SM, e.g., on the mass and the mixing angle of a possible right-handed W-boson. To confront the experimental results with SM, adequately accurate theoretical predictions should be provided. This necessarily requires to calculate radiative corrections within the perturbative Quantum Electrodynamics (QED). In this paper we will concentrate on the effect of higher order leading and next-to-leading logarithmic corrections.

2. Preliminaries

Within the Standard Model, the differential distribution of electrons (averaged over electron spin states) in the polarized muon decay reads

$$\frac{\mathrm{d}^2\Gamma^{\mu^\mp\to e^\mp\nu\bar{\nu}}}{\mathrm{d}x\mathrm{d}c} = \Gamma_0\left(F(x) \pm cP_\mu G(x)\right), \qquad \Gamma_0 = \frac{G_F^2 m_\mu^5}{192\pi^3},$$
$$c = \cos\theta, \qquad x = \frac{2m_\mu E_e}{m_\mu^2 + m_e^2}, \qquad x_0 \le x \le 1, \qquad x_0 = \frac{2m_\mu m_e}{m_\mu^2 + m_e^2}, \tag{2.1}$$

where m_μ and m_e are the muon and electron masses; G_F is the Fermi coupling constant; θ is the angle between the muon polarization vector $\vec{P}_\mu$ and the electron (or positron) momentum; E_e and x are the energy and the energy fraction of $e^\pm$. Here we adopt the definition of the Fermi coupling constant following ref. [7]. Within the Standard Model, the muon decay happens due to the weak interaction of leptons and W-bosons. The Fermi model corresponds to the limiting case of the infinite W-boson mass. If G_F is defined according to refs. [5, 6], the first order effect in the muon and W-boson mass ratio gives

$$\Gamma_0 \longrightarrow \Gamma_0\left(1 + \frac{3}{5}\frac{m_\mu^2}{m_W^2}\right). \tag{2.2}$$

Functions $F(x)$ and $G(x)$ describe the isotropic and anisotropic parts of the spectrum, respectively. Within perturbative QED, they can be expanded in series in the fine structure constant α:

$$F(x) = f_{\text{Born}}(x) + \frac{\alpha}{2\pi}f_1(x) + \left(\frac{\alpha}{2\pi}\right)^2 f_2(x) + \left(\frac{\alpha}{2\pi}\right)^3 f_3(x) + \mathcal{O}\left(\alpha^4\right), \tag{2.3}$$

and in the same way for $G(x)$. The Born-level functions f_{Born} and g_{Born} are well known, including small terms suppressed by the m_e/m_μ mass ratio [1]:

$$f_{\text{Born}}(x) = 6x\left(1 + \frac{m_e^2}{m_\mu^2}\right)^4\sqrt{1 - \frac{m_e^2}{E_e^2}}\left[x(1-x) + \frac{2}{9}\rho(4x^2 - 3x - x_0^2) + \eta x_0(1-x)\right],$$
$$g_{\text{Born}}(x) = -2x^2\xi\left(1 + \frac{m_e^2}{m_\mu^2}\right)^4\left(1 - \frac{m_e^2}{E_e^2}\right)\left[1 - x + \frac{2}{3}\delta\left(4x - 3 - \frac{m_e}{m_\mu}x_0\right)\right], \tag{2.4}$$

where ρ, η, ξ, and δ are the so-called Michel parameters [8, 9, 10], which in the Standard Model take the following values: $\rho = 3/4$, $\eta = 0$, $\xi = 1$, and $\delta = 3/4$. By fitting the values of the parameters from the experimental data and comparing them with the SM predictions, the $\mathcal{TWIST}$ experiment is going to look for effects of non-standard interactions.

The first order corrections $f_1(x)$ and $g_1(x)$ are also known with the exact account of the dependence on the electron mass [11, 12, 13]. Starting from $\mathcal{O}(\alpha)$, radiative corrections to the electron spectrum contain so-called mass singularities in the form of the large logarithm $L \equiv \ln(m_\mu^2/m_e^2) \approx 10.66$. As demonstrated in ref. [14] (see also table 1 below), the terms of the order $\mathcal{O}(\alpha L)$ give the bulk of the first order correction. This is our reason to look first for higher order terms enhanced by the large logarithm. These enhanced terms (excluding the ones related to the running of the QED coupling constant) cancel out in the expression for the total muon decay width at any order in α in accord with the Kinoshita-Lee-Nauenberg theorem [15, 16].

The second order corrections to the total muon decay width were calculated in ref. [17]. At this order for the differential decay spectrum, we know only the leading logarithmic (LL) corrections [14] and the isotropic part in the next-to-leading logarithmic (NLL) approximation [18]. The corresponding anisotropic part will be given below. The third order LL corrections will be presented here as well.

3. The fragmentation function approach

First, I will describe briefly the application of the renormalization group method to the calculation of the leading and next-to-leading radiative corrections to the polarized muon decay spectrum. For a detailed foundation of the procedure and an extended discussion see ref. [18].

The factorizations theorems, proved for QCD [19], can be easily translated to QED. In particular, by means of factorization, one can represent the differential spectrum of electrons as a convolution:

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}x\mathrm{d}c}(x,c,m_\mu,m_e) = \sum_{j=e,\gamma}\int_x^1 \frac{\mathrm{d}z}{z}\,\frac{\mathrm{d}^2\hat{\Gamma}_j}{\mathrm{d}z\mathrm{d}c}(z,c,m_\mu,\mu_f)\mathcal{D}_j\left(\frac{x}{z},\mu_f,m_e\right), \tag{3.1}$$

where $\mathrm{d}^2\hat{\Gamma}_j/(\mathrm{d}z\mathrm{d}c)$ is the differential distribution a muon decay with production of a massless electron ($j = e$) or a photon ($j = \gamma$) with energy fraction z ($z = 2E_j/m_\mu$, where E_j is the energy of the relevant particle). To subtract collinear singularities from the differential distributions, we will use here the $\overline{\mathrm{MS}}$ factorization scheme [20] with the scale μ_f. The fragmentation function $\mathcal{D}_j(x/z,\mu_f,m_e)$ describes the conversion of a massless parton j into a massive *physical* electron.

The spectrum of the massless parton can be expanded in a perturbative series:

$$\frac{1}{\Gamma_0}\frac{\mathrm{d}^2\hat{\Gamma}_j}{\mathrm{d}z\mathrm{d}c}(z,c,m_\mu,\mu_f) = \hat{A}_j^{(0)}(z,c)+\frac{\bar{\alpha}(\mu_f)}{2\pi}\hat{A}_j^{(1)}(z,c)+\left(\frac{\bar{\alpha}(\mu_f)}{2\pi}\right)^2\hat{A}_j^{(2)}(z,c)+\mathcal{O}\left(\alpha^3\right)\,, \tag{3.2}$$

where $\bar{\alpha}(\mu_f)$ is the $\overline{\mathrm{MS}}$ renormalized coupling constant, which will be converted further into the traditional QED on-shell coupling constant $\alpha \approx 1/137.036$. The lowest order spectrum of massless electrons is defined by

$$\hat{A}_e^{(0)}(z,c) = f_0(z) \pm cP_\mu g_0(z)\,, \qquad f_0(z) = z^2(3-2z)\,, \qquad g_0(z) = z^2(1-2z)\,. \tag{3.3}$$

Here and in what follows, the sign before c should be chosen according to the charge of the decaying muon (plus for μ^- decay and *vice versa*). The $\mathcal{O}(\alpha)$ correction to the massless electron spectrum reads

$$\hat{A}_e^{(1)}(z,c) \;=\; \hat{f}_e^{(1)}(z) \pm cP_\mu\hat{g}_e^{(1)}(z), \tag{3.4}$$

$$
\begin{aligned}
\hat{g}_e^{(1)}(z) \,=\, & \left(2z^2(1-2z)\ln\frac{1-z}{z}-\frac{1}{6}-4z^2+\frac{8}{3}z^3\right)\ln\frac{m_\mu^2}{\mu_f^2}+ \\
& +2z^2(1-2z)\Big(\ln^2(1-z)-4\mathrm{Li}_2\,(1-z)-\ln z\ln(1-z)-2\ln^2 z\Big)+ \\
& +\left(\frac{11}{3}-\frac{4}{3z}-6z-\frac{17}{3}z^2+\frac{34}{3}z^3\right)\ln(1-z)+\left(-\frac{1}{3}-6z^2-6z^3\right)\ln z- \\
& -\frac{7}{6}+3z+\frac{7}{6}z^2+3z^3\,. \qquad (3.5)
\end{aligned}
$$

For the auxiliary photon spectrum with collinear singularities subtracted according to the $\overline{\mathrm{MS}}$ prescription, we have

$$
\hat{A}_\gamma^{(0)}(z,c) \,=\, 0, \qquad \hat{A}_\gamma^{(1)}(z,c) \,=\, \hat{f}_\gamma^{(1)}(z) \pm cP_\mu \hat{g}_\gamma^{(1)}(z), \qquad (3.6)
$$

$$
\begin{aligned}
\hat{g}_\gamma^{(1)}(z) \,=\, & \left(\frac{1}{3}-\frac{1}{3z}-\frac{2}{3}z^2+\frac{2}{3}z^3\right)\left(\ln\frac{m_\mu^2}{\mu_f^2}+\ln(1-z)\right)+ \\
& +\left(\frac{2}{3}-\frac{2}{3z}\right)\ln z-\frac{2}{3}+\frac{2}{3z}+\frac{11}{12}z-\frac{2}{3}z^2-\frac{1}{4}z^3\,. \qquad (3.7)
\end{aligned}
$$

The isotropic parts of the first order corrections to the auxiliary massless parton distributions, $\hat{f}_e^{(1)}(z)$ and $\hat{f}_\gamma^{(1)}(z)$, are given by eqs. (7.8) in ref. [18]. By the choice of the factorization parameter value, $\mu_f \sim m_\mu$, we avoid calculation of the unknown functions $\hat{A}_j^{(2)}$, since then they can not give rise to any LL or NLL correction.

The fragmentation functions $\mathcal{D}_j$ can be obtained by solving the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations for QED,

$$
\frac{\mathrm{d}\mathcal{D}_i(x,\mu_f,m_e)}{\mathrm{d}\ln\mu_f^2} = \sum_{j=e,\gamma}\int_x^1 \frac{\mathrm{d}z}{z}\,P_{ji}(z,\bar{\alpha}(\mu_f))\mathcal{D}_j\left(\frac{x}{z},\mu_f,m_e\right), \qquad (3.8)
$$

where P_{ji} are perturbative splitting functions,

$$
P_{ji}(z,\bar{\alpha}(\mu_f)) = \frac{\bar{\alpha}(\mu_f)}{2\pi}P_{ji}^{(0)}(z)+\left(\frac{\bar{\alpha}(\mu_f)}{2\pi}\right)^2 P_{ji}^{(1)}(z)+\mathcal{O}\left(\alpha^3\right)\,. \qquad (3.9)
$$

The initial conditions, which are required to solve the DGLAP equations by iterations, can be borrowed from QCD studies [21]:

$$
\begin{aligned}
\mathcal{D}_e^{\mathrm{ini}}(x,\mu_0,m_e) \,&=\, \delta(1-x)+\frac{\bar{\alpha}(\mu_0)}{2\pi}d_1(x,\mu_0,m_e)+\mathcal{O}\left(\alpha^2\right), \\
d_1(x,\mu_0,m_e) \,&=\, \left[\frac{1+x^2}{1-x}\left(\ln\frac{\mu_0^2}{m_e^2}-2\ln(1-x)-1\right)\right]_+, \qquad (3.10) \\
\mathcal{D}_\gamma^{\mathrm{ini}}(x,\mu_0,m_e) \,&=\, \frac{\bar{\alpha}(\mu_0)}{2\pi}\ln\frac{\mu_0^2}{m_e^2}P_{\gamma e}^{(0)}(x)+\mathcal{O}\left(\alpha^2\right)\,. \qquad (3.11)
\end{aligned}
$$

The relevant lowest order splitting functions are

$$P^{(0)}_{ee}(x) = \left[\frac{1+x^2}{1-x}\right]_+ , \qquad P^{(0)}_{e\gamma}(x) = \frac{1+(1-x)^2}{x} , \qquad P^{(0)}_{\gamma e}(x) = x^2 + (1-x)^2 . \quad (3.12)$$

The *plus* prescription works as usually:

$$\int_{x_{\min}}^{1} \mathrm{d}x\ [V(x)]_+ W(x) = \int_0^1 \mathrm{d}x\, V(x)[W(x)\Theta(x - x_{\min}) - W(1)] ,$$
$$\Theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} . \quad (3.13)$$

In a measurement of the muon decay spectrum, events with more than one electron in the final state require a special treatment. Starting from the second order in α, we have a certain contribution due to emission of real and virtual e^+e^- pairs. Presumably a Monte Carlo event generator is needed to simulate the process of muon decay with pair production. Nevertheless, we will calculate the corresponding effect under a simple assumption, that an event with two electrons in the final state is treated as a pair of simultaneous muon decays. In order to have the possibility to drop the pair contributions (if they are taken into account in a Monte Carlo program), we decompose our results according to the classes of the corresponding Feynman diagrams in the same way as in ref. [22]. Moreover, the decomposition will help us to demonstrate the cancellation of the mass singularities in the integrated decay width. Our results for pure pair corrections can serve further also as a benchmark for the Monte Carlo program. The next-to-leading electron splitting function can be decomposed into four parts:

$$P^{(1)}_{ee}(x) = P^{(1,\gamma)}_{ee}(x) + P^{(1,\mathrm{NS})}_{ee}(x) + P^{(1,\mathrm{S})}_{ee}(x) + P^{(1,\mathrm{int})}_{ee}(x) , \quad (3.14)$$

where $P^{(1,\gamma)}_{ee}(x)$ is provided by the set of Feynman diagrams with pure photonic corrections; $P^{(1,\mathrm{NS})}_{ee}(x)$ is related to the corrections due to non-singlet real and virtual pairs; $P^{(1,\mathrm{S})}_{ee}(x)$ stands for the singlet pair production contribution; and $P^{(1,\mathrm{int})}_{ee}(x)$ describes the interference of real singlet and non-singlet pairs. By extracting the appropriate color structures from the known QCD results [23, 24, 25, 26], the explicit expressions for these functions were given in ref. [18]. Here and in what follows, in the language of Feynman diagrams, the situation when the registered electron is connected by a solid fermion line with the genuine electroweak decay vertex is called *non-singlet*. The case, when the observed electron belongs to a pair produced via a virtual photon, is called *singlet*.

The relation between the $\overline{\mathrm{MS}}$ and the on-shell coupling constants reads [7]

$$\bar{\alpha}(\mu_f) = \alpha + \frac{\alpha^2}{3\pi}\ln\frac{\mu_f^2}{m_e^2} + \frac{\alpha^3}{4\pi^2}\ln\frac{\mu_f^2}{m_e^2} + \frac{15\alpha^3}{16\pi^2} + \mathcal{O}\left(\alpha^4\right) . \quad (3.15)$$

It is convenient to choose the renormalization scale to be

$$\mu_0 = m_e . \quad (3.16)$$

Now we have everything for solving the DGLAP equations (3.8). Using iterations for the electron fragmentation function decomposed into four parts, we get

$$\begin{aligned}
\mathcal{D}_e(x,\mu_f,m_e) &= \mathcal{D}_e^{(\gamma)}(x) + \mathcal{D}_e^{(\mathrm{NS})}(x) + \mathcal{D}_e^{(\mathrm{S})}(x) + \mathcal{D}_e^{(\mathrm{int})}(x)\,, && (3.17)\\
\mathcal{D}_e^{(\gamma)}(x) &= \delta(1-x) + \frac{\alpha}{2\pi}d_1(x,\mu_0,m_e) + \frac{\alpha}{2\pi}L_f P_{ee}^{(0)}(x) + \\
&\quad + \left(\frac{\alpha}{2\pi}\right)^2\left(\frac{1}{2}L_f^2 P_{ee}^{(0)}\otimes P_{ee}^{(0)}(x) + L_f P_{ee}^{(0)}\otimes d_1(x,\mu_0,m_e) + L_f P_{ee}^{(1,\gamma)}(x)\right) + \\
&\quad + \left(\frac{\alpha}{2\pi}\right)^3\frac{1}{6}L_f^3 P_{ee}^{(0)}\otimes P_{ee}^{(0)}\otimes P_{ee}^{(0)}(x)\,, && (3.18)\\
\mathcal{D}_e^{(\mathrm{NS})}(x) &= \left(\frac{\alpha}{2\pi}\right)^2\left(\frac{1}{3}L_f^2 P_{ee}^{(0)}(x) + L_f P_{ee}^{(1,\mathrm{NS})}(x)\right) + \\
&\quad + \left(\frac{\alpha}{2\pi}\right)^3 L_f^3\left(\frac{1}{3}P_{ee}^{(0)}\otimes P_{ee}^{(0)}(x) + \frac{4}{27}P_{ee}^{(0)}(x)\right), && (3.19)\\
\mathcal{D}_e^{(\mathrm{S})}(x) &= \left(\frac{\alpha}{2\pi}\right)^2\left(\frac{1}{2}L_f^2 P_{e\gamma}^{(0)}\otimes P_{\gamma e}^{(0)}(x) + L_f P_{ee}^{(1,\mathrm{S})}(x)\right) + \\
&\quad + \left(\frac{\alpha}{2\pi}\right)^3 L_f^3\left(\frac{1}{3}P_{e\gamma}^{(0)}\otimes P_{\gamma e}^{(0)}\otimes P_{ee}^{(0)}(x) - \frac{1}{9}P_{e\gamma}^{(0)}\otimes P_{\gamma e}^{(0)}(x)\right), && (3.20)\\
\mathcal{D}_e^{(\mathrm{int})}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 L_f P_{ee}^{(1,\mathrm{int})}(x)\,, \qquad L_f \equiv \ln\frac{\mu_f^2}{m_e^2}\,, && (3.21)
\end{aligned}$$

where we systematically omitted terms of the following orders: $\mathcal{O}\left(\alpha^2 L_f^0\right)$, $\mathcal{O}\left(\alpha^3 L_f^2\right)$, $\mathcal{O}\left(\alpha^4\right)$, and higher. The photon fragmentation function at the lowest order,

$$\mathcal{D}_\gamma(x,\mu_f,m_e) = \frac{\alpha}{2\pi}L_f P_{e\gamma}^{(0)}(x) + \mathcal{O}\left(\alpha^2\right)\,, \tag{3.22}$$

is sufficient for our purposes. The convolution operation is defined in the standard way:

$$A\otimes B(x) = \int_0^1 \mathrm{d}z\int_0^1 \mathrm{d}z'\,\delta(x-zz')A(z)B(z') = \int_x^1\frac{\mathrm{d}z}{z}\,A(z)B\left(\frac{x}{z}\right). \tag{3.23}$$

The leading logarithmic terms for the QED fragmentation function are known up to the fifth order in α [27]. But, as will be seen from the numerical results, keeping contributions up to the third order is enough for the moment.

The explicit expressions for the functions, which appear in the $\mathcal{O}\left(\alpha^2\right)$ terms of the function $\mathcal{D}_e$, are given in ref. [18]. In the third order we have two more functions [28]:

$$\begin{aligned}
P_{ee}^{(0)}\otimes P_{ee}^{(0)}\otimes P_{ee}^{(0)}(x) &= \delta(1-x)\left(16\zeta_3 - \frac{81}{4}\right) + \\
&\quad + \left[24\frac{1+x^2}{1-x}\left(\frac{1}{2}\ln^2(1-x) + \frac{3}{4}\ln(1-x) + \frac{9}{32} - \frac{1}{2}\zeta_2\right)\right]_+ \\
&\quad + 24\frac{1+x^2}{1-x}\left(\frac{1}{12}\ln^2 x - \frac{1}{2}\ln x\ln(1-x) - \frac{3}{8}\ln x\right) + \\
&\quad + 6(1+x)\ln x\ln(1-x) - 12(1-x)\ln(1-x) + \frac{3}{2}(5-3x)\ln x - \\
&\quad - 3(1-x) - \frac{3}{2}(1+x)\ln^2 x + 6(1+x)\mathrm{Li}_2\left(1-x\right)\,, && (3.24)
\end{aligned}$$

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$$P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)} \otimes P_{ee}^{(0)}(x) = (1+x)\Big(4\ln(1-x)\ln x - \ln^2 x + 4\mathrm{Li}_2\,(1-x)\Big) + \qquad (3.25)$$
$$+\frac{2}{3}(3x+4x^2)\ln x + \frac{2}{3}\left(\frac{4}{x}+3-3x-4x^2\right)\ln(1-x) - \frac{23}{6}(1-x)\,,$$

where

$$\zeta_n \equiv \sum_{k=1}^{\infty}\frac{1}{k^n}\,, \qquad \zeta_2 = \frac{\pi^2}{6}\,, \qquad \mathrm{Li}_2\,(x) \equiv -\int_0^x \mathrm{d}y\,\frac{\ln(1-y)}{y}\,. \qquad (3.26)$$

By convolution of the fragmentation functions [eqs. (3.17) and (3.22)] with the differential distributions (3.2), we receive higher order corrections to the electron spectrum, which will be presented in section 5. In the results we can fix (see discussion in ref. [18]) the factorization scale

$$\mu_f = m_\mu \qquad (3.27)$$

and get $L_f \to L$.

4. Exponentiation

Looking at the end point of the energy spectrum ($x \to 1$) of unpolarized muon decay, one can recognize that the first order correction becomes there negative and very large, making the result senseless. An extensive discussion of the phenomenon can be found in refs. [12, 29]. The divergence is a clear signal to look beyond the first order approximation. In fact, the Yennie-Frautschi-Suura theorem [30] allows to make a re-summation of the dangerous terms and to convert them into a definitely positive exponent. The exponentiation procedure is not unique, it permits to involve also some terms convergent at $x \to 1$. In our case, the exponentiation is allowed to add several terms of the following types: $\mathcal{O}\left(\alpha^2 L^0\right)$, $\mathcal{O}\left(\alpha^n L^m\right)$ with $n \geq 3$, $0 \leq m < n$, and $\mathcal{O}\left(\alpha^n L^n\right)$, $n \geq 4$.

Let us consider two ways of exponentiation. In the first case one starts from the corrected cross section and tries to perform a re-summation of the known terms, which are divergent at $x \to 1$, by converting them into an exponent:

$$\left.\frac{F(x)}{f_0(x)}\right|_{x\to 1} \approx 1 + \frac{\alpha}{\pi}(L-2)\ln(1-x) + \cdots \longrightarrow \exp\left\{\frac{\alpha}{\pi}(L-2)\ln(1-x)\right\}. \qquad (4.1)$$

This is a kind of the so-called *ad hoc* exponentiation. The effect (see table 1) of this approach can be represented by the relative contribution of *new* terms generated by the exponent,

$$\delta^{\mathrm{exp}}_{\mathrm{a.h.}}(x) = \exp\left\{\frac{\alpha}{\pi}(L-2)\ln(1-x)\right\} - 1 - \frac{\alpha}{\pi}(L-2)\ln(1-x) -$$
$$-\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2(L^2-4L)\ln^2(1-x) - \frac{1}{6}\left(\frac{\alpha}{\pi}L\ln(1-x)\right)^3. \qquad (4.2)$$

The most significant term above is of the order $\alpha^2\ln^2(1-x)$ and gives a numerically important contribution at large x. Note that all the subtracted terms do appear in the perturbative results.

The next step should be to check that in higher orders the exponent doesn't contradict the known (or anticipated) results. The above procedure in the case of muon decay can be criticized [31], because the higher order leading logarithmic terms represent a mass singularity: one can not guarantee that all the large logarithms, coming from $\delta^{\mathrm{exp}}_{\mathrm{a.h.}}(x)$, disappear after the integration over the energy fraction. Nevertheless, the *ad hoc* exponentiation is not supposed to produce a complete result. The region of its application is limited: it deals with the terms, which are the most important in the end of the spectrum.

There is another way of exponentiation, which avoids the problem of improper mass singularities. One can use the exponentiated representation of the electron structure (fragmentation) function suggested in ref. [32], which obeys the proper normalization:

$$\int_0^1 \mathcal{D}^{(\gamma)}_{\mathrm{exp}}(x)\mathrm{d}x = 1\,. \tag{4.3}$$

The exponentiated structure functions are based on the exact solutions of the QED evolution equations in the limiting case of soft radiation. For computations I used an advanced formula from ref. [33], where I substituted $(L-1)$, which was natural for e^+e^- annihilation, by $(L-2)$. This substitution has not been obvious from the beginning, but it is clearly seen in the above *ad hoc* exponentiation. The usual $(L-1)$ factor corresponds to soft radiation off the electron, while the additional (-1) is due to radiation off the muon. In fact one can introduce the muon structure function into the master equation (3.1), as discussed in ref. [34]. The muon structure function does not give any large logarithms in our case. But still it can be used to describe the contribution of soft photon radiation off the muon. And the corresponding factor is just that (-1) instead of the usual $(L-1)$. We see, that it is easy to make a mistake in exponentiation of soft gluons (photons) in decay processes by forgetting about the radiation off the decaying particle.

The relevant for us electron structure function [33] is taken within the leading logarithm approximation for pure photonic corrections with terms up to $\mathcal{O}\left(\alpha^3 L^3\right)$ and supplied with exponentiation of some terms in higher orders. Convolution with the Born-level functions gives

$$F^{\mathrm{exp}}_{\mathrm{SF}}(x) = \mathcal{D}^{(\gamma)}_{\mathrm{exp}} \otimes f_0(x)\,, \qquad G^{\mathrm{exp}}_{\mathrm{SF}}(x) = \mathcal{D}^{(\gamma)}_{\mathrm{exp}} \otimes g_0(x)\,. \tag{4.4}$$

A subtraction of the known terms in the lower orders ($n \leq 3$) of the perturbative expansion, similar to eq. (4.2), is used to receive the value of the relative contribution $\delta^{\mathrm{exp}}_{\mathrm{SF}}(x)$. The latter contains also terms of the order $\mathcal{O}\left(\alpha^2 L^0\right)$, which are not singular at $x \to 1$. This is due to the fact that soft radiation is allowed not only at the end of the energy spectrum, but in any other point in x. The resulting effect is spread all over the spectrum and appears to be visible at the two ends $x \to 0$ and $x \to 1$. Numerical results (see table 1) of the two approaches are close to each other in the large-x region (where an analytical agreement between the approaches can be observed as well). Contrary to the exponentiation of soft gluons in QCD [34], our procedure doesn't suffer from the renormalization scheme (and scale) dependence and from non-perturbative effects.

Simultaneous exponentiation of photonic and pair corrections can be constructed as well. But it was criticized [35], since soft pairs (contrary to soft photons) have a non-zero production threshold, which can't be taken into account by exponentiation properly.

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5. Results

To the best of our present knowledge, we can write now function $F(x)$ from the master formula (2.1) as follows:

$$\begin{aligned}F(x) &= f_{\mathrm{Born}}(x) + \frac{\alpha}{2\pi}f_1(x) + \\ &+\left(\frac{\alpha}{2\pi}\right)^2\left\{\left[f_2^{(0,\gamma)}(x) + \frac{2}{3}f_2^{(0,\mathrm{NS})}(x) + f_2^{(0,\mathrm{S})}(x)\right]\frac{L^2}{2} + \right.\\ &\left.\qquad + \left[f_2^{(1,\gamma)}(x) + f_2^{(1,\mathrm{NS})}(x) + f_2^{(1,\mathrm{S})}(x) + f_2^{(1,\mathrm{int})}(x)\right]L\right\} + \\ &+\left(\frac{\alpha}{2\pi}\right)^3\left[f_3^{(0,\gamma)}(x) + f_3^{(0,\mathrm{NS})}(x) + f_3^{(0,\mathrm{S})}(x)\right]\frac{L^3}{6} + \Delta f_{\mathrm{exp}}(x) + \\ &+\mathcal{O}\left(\alpha^2L^0, \alpha^3L^2, \alpha^4L^4\right) . \end{aligned} \tag{5.1}$$

Function $G(x)$ takes the same form with the substitution $f_i \to g_i$. Effects due to virtual hadronic, $\mu^+\mu^-$, and $\tau^+\tau^-$ pairs [36] are not shown explicitly, since they are of the order $\mathcal{O}\left(\alpha^2L^0\right)$. The effect of exponentiation is given by

$$\Delta f_{\mathrm{exp}}(x) = f_0(x)\delta_{\mathrm{SF}}^{\mathrm{exp}}(x) , \qquad \Delta g_{\mathrm{exp}}(x) = g_0(x)\delta_{\mathrm{SF}}^{\mathrm{exp}}(x) . \tag{5.2}$$

5.1 Analytical results

In the second order for the NLL corrections to the anisotropic part of the electron energy distribution, we have

$$\begin{aligned}g_2^{(1,\gamma)}(x) &= 4x^2(1-2x)\Big(\mathrm{Li}_3(1-x) + \mathrm{S}_{1,2}(1-x) - 2\mathrm{Li}_2(1-x)\ln(1-x) + \\ &\qquad + \ln x\ln^2(1-x) - 3\ln^2 x\ln(1-x) + \ln^3 x - \zeta_2\ln x + \frac{3}{2}\zeta_3\Big) \\ &+\left(\frac{14}{3} - \frac{8}{3x} - 6x + 24x^2 - \frac{92}{3}x^3\right)\mathrm{Li}_2(1-x) + \left(6x - 5 - \frac{86}{3}x^2\right)\ln x\ln(1-x) + \\ &+\left(8 - \frac{8}{3x} - 12x + \frac{20}{3}x^2 + 8x^3\right)\ln^2(1-x) + \left(\frac{5}{12} + 18x^2 - \frac{70}{3}x^3\right)\ln^2 x + \\ &+\left(-\frac{13}{3} + \frac{37}{3}x + \frac{50}{3}x^2 - \frac{32}{3}x^3\right)\ln(1-x) + \left(\frac{25}{12} - \frac{59}{6}x + 6x^2 + \frac{32}{9}x^3\right)\ln x + \\ &+\left(-8 + \frac{8}{3x} + 12x - \frac{29}{3}x^2 - 2x^3\right)\zeta_2 + \frac{817}{216} - \frac{91}{12}x + \frac{62}{3}x^2 - \frac{607}{54}x^3 , \end{aligned} \tag{5.3}$$

$$\begin{aligned}g_2^{(1,\mathrm{NS})}(x) &= 4x^2(1-2x)\left(-\mathrm{Li}_2(1-x) - \frac{1}{3}\ln x\ln(1-x) + \frac{1}{3}\ln^2(1-x) - \frac{1}{2}\ln^2 x - \frac{1}{3}\zeta_2\right) + \\ &+\left(\frac{22}{9} - \frac{8}{9x} - 4x - 6x^2 + 12x^3\right)\ln(1-x) + \\ &+\left(-\frac{1}{9} + \frac{8}{9}x^2 - \frac{76}{9}x^3\right)\ln x - \frac{7}{18} + \frac{5}{3}x + \frac{86}{9}x^2 - \frac{20}{3}x^3 , \end{aligned} \tag{5.4}$$

$$g_2^{(1,\mathrm{S})}(x) = \Big(\mathrm{Li}_2(1-x) + \ln x\ln(1-x)\Big)\left(\frac{4}{3}x^2 - \frac{1}{3}\right) + \left(\frac{4}{3}x^2 - \frac{1}{2}\right)\ln^2 x +$$

$$
+\left(-\frac{1}{9}-\frac{2}{9x}+x+\frac{2}{9}x^2-\frac{8}{9}x^3\right)\ln(1-x)+\left(\frac{5}{9}-\frac{4}{9x}+\frac{5}{2}x+\frac{5}{9}x^2\right)\ln x+
$$

$$
+\frac{1}{3x}+\frac{4}{3}-\frac{7}{18}x-\frac{43}{18}x^2+\frac{10}{9}x^3\,, \tag{5.5}
$$

$$
\begin{aligned}
g_2^{(1,\mathrm{int})}(x) &= 4x^2(1-2x)\Big(\mathrm{Li}_3\,(1-x)-2\mathrm{S}_{1,2}\,(1-x)-\mathrm{Li}_2\,(1-x)\ln x\Big)+\\
&\quad+\left(-\frac{1}{3}-14x^2+\frac{52}{3}x^3\right)\mathrm{Li}_2\,(1-x)+\left(-3x^2+\frac{26}{3}x^3\right)\ln^2 x+\\
&\quad+\left(\frac{1}{3}+\frac{1}{3}x-\frac{28}{3}x^2\right)\ln x+\frac{10}{9}-\frac{1}{3}x-\frac{37}{3}x^2+\frac{104}{9}x^3\,.
\end{aligned} \tag{5.6}
$$

The polylogarithm functions are defined as

$$
\mathrm{Li}_3\,(x)\equiv\int_0^x \mathrm{d}y\,\frac{\mathrm{Li}_2\,(y)}{y}\,,\qquad \mathrm{S}_{1,2}\,(x)\equiv\frac{1}{2}\int_0^x \mathrm{d}y\,\frac{\ln^2(1-y)}{y}\,. \tag{5.7}
$$

The $\mathcal{O}\left(\alpha^2L^2\right)$ corrections $f_2^{(0,j)}(x)$ and $g_2^{(0,j)}(x)$ $(j=\gamma$, NS, S) can be found in ref. [14]. Explicit expressions for the second order next-to-leading corrections to the isotropic part of the spectrum $(f_2^{(1,i)}(x),\ i=\gamma$, NS, S, int) are given in ref. [18].

The third order LL photonic contributions read

$$
\begin{aligned}
f_3^{(0,\gamma)}(x) &= 8x^2(3-2x)\Psi(x)+(10+24x-48x^2+32x^3)\ln^2(1-x)+\\
&\quad+\left(\frac{5}{12}+x-8x^2+16x^3\right)\ln^2 x+(-5-12x+48x^2-64x^3)\ln x\ln(1-x)+\\
&\quad+(5+12x-32x^3)\mathrm{Li}_2\,(1-x)+(-10-24x+48x^2-32x^3)\zeta_2+\\
&\quad+\left(-\frac{13}{18}-\frac{21}{2}x+\frac{64}{3}x^3\right)\ln x+\left(\frac{11}{6}+17x+16x^2-\frac{64}{3}x^3\right)\ln(1-x)+\\
&\quad+\frac{569}{216}+\frac{4}{3}x-\frac{16}{3}x^2+\frac{128}{27}x^3\,,
\end{aligned} \tag{5.8}
$$

$$
\begin{aligned}
g_3^{(0,\gamma)}(x) &= 8x^2(1-2x)\Psi(x)+(-2-48x^2+32x^3)\ln^2(1-x)+\\
&\quad+\left(-\frac{1}{12}-8x^2+16x^3\right)\ln^2 x+(1+48x^2-64x^3)\ln x\ln(1-x)+\\
&\quad+(-1-32x^3)\mathrm{Li}_2\,(1-x)+(2+48x^2-32x^3)\zeta_2+\\
&\quad+\left(\frac{5}{18}+\frac{5}{2}x+\frac{64}{3}x^3\right)\ln x+\left(-\frac{7}{6}-7x+16x^2-\frac{64}{3}x^3\right)\ln(1-x)-\\
&\quad-\frac{133}{216}-\frac{13}{6}x-\frac{16}{3}x^2+\frac{128}{27}x^3\,,
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
\Psi(x) &\equiv 3\mathrm{Li}_3\,(1-x)-2\mathrm{S}_{1,2}\,(1-x)+\ln^3(1-x)-\frac{1}{6}\ln^3 x+\frac{3}{2}\ln^2 x\ln(1-x)-\\
&\quad-3\ln x\ln^2(1-x)-3\mathrm{Li}_2\,(1-x)\ln(1-x)+2\zeta_3-3\zeta_2\ln\frac{1-x}{x}\,.
\end{aligned} \tag{5.10}
$$

And the third order LL pair corrections[1] are

$$
\begin{aligned}
f_3^{(0,\mathrm{NS})}(x) &= 8x^2(3-2x)\Phi(x) + \left(\frac{20}{3} + 16x - \frac{80}{3}x^2 + \frac{160}{9}x^3\right)\ln(1-x) + \\
&+\left(-\frac{5}{3} - 4x + \frac{32}{3}x^2 - \frac{160}{9}x^3\right)\ln x + \frac{73}{54} + \frac{67}{9}x + \frac{16}{9}x^2 - \frac{128}{27}x^3, \qquad (5.11)\\
g_3^{(0,\mathrm{NS})}(x) &= 8x^2(1-2x)\Phi(x) + \left(-\frac{4}{3} - \frac{272}{9}x^2 + \frac{160}{9}x^3\right)\ln(1-x) + \\
&+\left(\frac{1}{3} + \frac{128}{9}x^2 - \frac{160}{9}x^3\right)\ln x - \frac{29}{54} - \frac{7}{3}x + \frac{16}{9}x^2 - \frac{128}{27}x^3\,, \qquad (5.12)\\
f_3^{(0,\mathrm{S})}(x) &= \left(\frac{5}{3} + 4x + 4x^2\right)\left(4\mathrm{Li}_2\,(1-x) + 4\ln x\ln(1-x) - \ln^2 x\right) - 4x^2\ln^2 x + \\
&+\left(\frac{68}{9} + \frac{8}{3x} + 12x - \frac{56}{3}x^2 - \frac{32}{9}x^3\right)\ln(1-x) + \qquad (5.13)\\
&+\left(-\frac{29}{9} - \frac{14}{3}x + 16x^2 + \frac{32}{9}x^3\right)\ln x - \frac{287}{27} - \frac{4}{9x} - \frac{13}{9}x + \frac{86}{9}x^2 + \frac{80}{27}x^3\,,\\
g_3^{(0,\mathrm{S})}(x) &= \left(\frac{4}{3}x^2 - \frac{1}{3}\right)\left(4\mathrm{Li}_2\,(1-x) + 4\ln x\ln(1-x) - \ln^2 x\right) - \frac{4}{3}x^2\ln^2 x + \\
&+\left(-\frac{4}{9} - \frac{8}{9x} + 4x + \frac{8}{9}x^2 - \frac{32}{9}x^3\right)\ln(1-x) + \\
&+\left(\frac{1}{9} - 2x - \frac{16}{9}x^2 + \frac{32}{9}x^3\right)\ln x + \frac{31}{27} + \frac{4}{27x} - \frac{35}{9}x - \frac{10}{27}x^2 + \frac{80}{27}x^3\,, \qquad (5.14)\\
\Phi(x) &\equiv \frac{1}{2}\ln^2 x + \ln^2(1-x) - 2\ln x\ln(1-x) - \mathrm{Li}_2\,(1-x) - \zeta_2\,. \qquad (5.15)
\end{aligned}
$$

Functions $g_2^{(1,\mathrm{S})}$ and $g_2^{(1,\mathrm{int})}$, shown above, as well as $f_2^{(1,\mathrm{S})}$ and $f_2^{(1,\mathrm{int})}$, which are given in ref. [18], coincide with the results of my calculations starting directly from Feynman diagrams and using methods described in ref. [37].

5.2 Cancellation of mass singularities

An important check of the results is to demonstrate the cancellation of mass singularities in the total decay width. The cancellation of the mass singularities in the LL contributions due to photons and non-singlet pairs is rather trivial:

$$
\int_0^1 \mathrm{d}x\; f_n^{(0,j)}(x) = \int_0^1 \mathrm{d}x\; g_n^{(0,j)}(x) = 0\,, \qquad j=\mathrm{NS},\gamma\,, \qquad n = 1,2,3,\ldots \qquad (5.16)
$$

It is guaranteed by the normalization conditions of the corresponding LL fragmentation functions.

Now we should note that a naive integration of the electron spectrum gives rather the counting rate of electrons than the total muon decay width, since the number of the final state electrons can exceed the number of decaying muons because of real e^+e^- pair emission. In other words, we should avoid the double counting of electrons in the contributions due to real pair emission. For this purpose, we can keep the non-singlet pair contributions and drop the singlet ones (see their definition on page 5).

[1]Strictly speaking, we have here pair and photonic corrections simultaneously.

Functions $f_2^{(1,\mathrm{int})}$ and $g_2^{(1,\mathrm{int})}$ contain the double counting too. To resolve this problem we can use the splitting function [23, 38]

$$P_{\bar{e}e}^{(1,\mathrm{int})}(x) = 2\frac{1+x^2}{1+x}\left(-2\mathrm{Li}_2\left(-x\right) - 2\ln x \ln(1+x) + \frac{1}{2}\ln^2 x - \zeta_2\right) + \\ +2(1+x)\ln x + 4(1-x)\,, \tag{5.17}$$

which describes the transition of an electron into a positron in the relevant set of Feynman diagrams. The corresponding contribution can be constructed by convolution with the lowest order functions f_0 and g_0 integrated over the positron energy fraction. One can see now that in the interference contribution the number of electrons is really twice as large as the number of positrons:

$$\int_0^1 \mathrm{d}x f_2^{(1,\mathrm{int})}(x) = 2\int_0^1 \mathrm{d}x P_{\bar{e}e}^{(1,\mathrm{int})} \otimes f_0(x) = \frac{13}{4} - 3\zeta_2 + 2\zeta_3\,. \tag{5.18}$$

By resolving the problem of double counting in the contribution of the S-NS pair interference, we arrive to the cancellation of mass singularities in the following form:

$$\int_0^1 \mathrm{d}x \left(f_2^{(1,\gamma)}(x) + f_2^{(1,\mathrm{int})}(x) - P_{\bar{e}e}^{(1,\mathrm{int})} \otimes f_0(x)\right) = \int_0^1 \mathrm{d}x \left(f_2^{(1,\gamma)}(x) + \frac{1}{2} f_2^{(1,\mathrm{int})}(x)\right) \\ = 0\,. \tag{5.19}$$

Let us look now at the integral of the second order NLL non-singlet pair correction. It is known [17], that the integrated contribution of this correction contains large logarithms due to the running of the coupling constant:

$$\left(\frac{\alpha}{2\pi}\right)^2 L \int_0^1 \mathrm{d}x f_2^{(1,\mathrm{NS})}(x) = \frac{\Delta\alpha(m_\mu)}{2\pi}\left[\int_0^1 \mathrm{d}x\ f_1(x)\right]\Bigg|_{m_e\to 0}, \tag{5.20}$$

where

$$\Delta\alpha(m_\mu) = \frac{\alpha^2}{3\pi}L = \alpha(m_\mu) - \alpha + \mathcal{O}\left(\alpha^2 L^0\right)\,. \tag{5.21}$$

To demonstrate that using our results, note first that the relevant function consists of two parts:

$$f_2^{(1,\mathrm{NS})}(x) = P_{ee}^{(1,\mathrm{NS})} \otimes f_0(x) + \frac{2}{3}\hat{f}_1(x)\,. \tag{5.22}$$

One can check that

$$\int_0^1 \mathrm{d}x P_{ee}^{(1,\mathrm{NS})} \otimes f_0(x) = 0\,. \tag{5.23}$$

It remains now to recognize that

$$\left[\int_0^1 \mathrm{d}x f_1(x)\right]\Bigg|_{m_e\to 0} = \int_0^1 \mathrm{d}x \hat{f}_1(x)\,, \tag{5.24}$$

which can be verified easily. Thus we checked successfully an important property of our analytical results.

The anisotropic contributions to the decay spectrum can be treated in the same way. They don't contribute to the total decay width at all, because they vanish after the integration over the angle. Nevertheless, the cancellation of mass singularities can be observed also in the forward-backward asymmetry of the decay, which is not affected by isotropic functions on the contrary. In particular, we have an equality analogous to eq. (5.18):

$$\int_0^1 \mathrm{d}x g_2^{(1,\mathrm{int})}(x) = 2\int_0^1 \mathrm{d}x P_{\bar{e}e}^{(1,\mathrm{int})} \otimes g_0(x) = -\frac{13}{12} + \zeta_2 - \frac{2}{3}\zeta_3 \,. \qquad (5.25)$$

5.3 Numerical results

Now we can make numerical estimates of the effects due to higher order radiative corrections. We should confront them with the $1\cdot 10^{-4}$ precision level of the $\mathcal{TWIST}$ experiment. In a typical experiment on muon decays one has almost 100% longitudinal polarization of muons ($P_\mu = 1$), since the muons are coming polarized from pion decays. In order to extract information about the Michel parameters ξ and δ one should study carefully the angular distribution of the electrons. But as can be seen from the analytical formulae, the dependence on the angle is rather simple and smooth. For this reason we restrict our numerical illustrations to the consideration of x-dependence at several fixed values of the angle.

Numerical results for the pure photonic corrections are presented in table 1, where we give the values of different contributions normalized by the Born distribution in the following way:

$$\begin{aligned}
\delta_1 &= \frac{\alpha}{2\pi}\,\frac{f_1(x) \pm cP_\mu g_1(x)}{f_{\mathrm{Born}}(x) \pm cP_\mu g_{\mathrm{Born}}(x)}\,, \\
\delta_n^{(0,\gamma)} &= \frac{L^n}{n!}\left(\frac{\alpha}{2\pi}\right)^n \frac{f_n^{(0,\gamma)}(x) \pm cP_\mu g_n^{(0,\gamma)}(x)}{f_{\mathrm{Born}}(x) \pm cP_\mu g_{\mathrm{Born}}(x)}\,, \qquad n = 1,2,3\,, \\
\delta_2^{(1,\gamma)} &= L\left(\frac{\alpha}{2\pi}\right)^2 \frac{f_2^{(1,\gamma)}(x) \pm cP_\mu g_2^{(1,\gamma)}(x)}{f_{\mathrm{Born}}(x) \pm cP_\mu g_{\mathrm{Born}}(x)}\,. \qquad (5.26)
\end{aligned}$$

One can see that the first order LL correction $\delta_1^{(0,\gamma)}$ (look for $f_1^{(0,\gamma)}(x)$ and $g_1^{(0,\gamma)}(x)$ in ref. [14]) provides the bulk of the effect, especially in the region of intermediate and large x-values. Convergence of the corresponding series in L in the second order corrections doesn't look so good: the NLL contribution is only about two times less than the LL one.

There is a trick, which allows to make a better approximation in the region of small x. Looking closely at the argument of the large logarithm during the actual calculations of integrals over the phase space of real photons, one can notice that it is rather $x^2 m_\mu^2/m_e^2$, than simply m_μ^2/m_e^2. So the modification $L \to L + 2\ln x$ can be done in our formulae. This will move some terms from the sub-leading corrections into the leading ones. I checked that the trick does really help to improve the agreement in the first order between $\delta_1^{(0,\gamma)}$ and the full δ_1. But, as far as the $\mathcal{TWIST}$ experiment is interested in the region $x \geq 0.3$ (the event distribution is peaked at large x-values in any case), I don't apply the modification here.

x	$10^4\delta_1$	$10^4\delta_1^{(0,\gamma)}$	$10^4\delta_2^{(0,\gamma)}$	$10^4\delta_2^{(1,\gamma)}$	$10^4\delta_3^{(0,\gamma)}$	$10^4\delta_{\mathrm{a.h.}}^{\mathrm{exp}}$	$10^4\delta_{\mathrm{SF}}^{\mathrm{exp}}$
0.05	4590.1	10325.0	184.96	−247.63	−0.43	0.00	6.90
0.1	1715.1	3257.7	33.18	−37.79	−0.35	0.00	1.39
0.2	674.0	1106.2	−1.28	0.34	−0.15	0.01	0.05
0.3	364.0	549.1	−6.58	4.48	−0.05	0.01	−0.20
0.5	64.1	82.6	−6.50	3.73	0.05	0.06	−0.26
0.7	−160.3	−214.9	−1.28	0.70	0.07	0.18	−0.09
0.9	−470.3	−592.1	15.19	−5.97	−0.14	0.72	0.62
0.99	−971.9	−1198.8	69.84	−26.08	−1.95	3.47	3.70
0.999	−1439.8	−1772.5	155.10	−57.86	−6.64	9.17	9.75

Table 1: Photonic corrections to μ^- decay spectrum versus x for $c = 1$ and $P_\mu = 1$.

	$c = 1$			$c = -1$		
x	$10^4\delta_2^{(0,e\bar{e})}$	$10^4\delta_2^{(1,e\bar{e})}$	$10^4\delta_3^{(0,e\bar{e})}$	$10^4\delta_2^{(0,e\bar{e})}$	$10^4\delta_2^{(1,e\bar{e})}$	$10^4\delta_3^{(0,e\bar{e})}$
0.05	548.14	−661.94	−2.19	2132.14	−2848.25	−9.12
0.1	66.02	−68.36	−0.18	241.62	−274.34	−0.76
0.2	9.55	−9.99	−0.05	29.25	−29.68	−0.04
0.3	3.43	−3.58	−0.06	9.59	−9.77	−0.05
0.5	0.49	−0.71	−0.06	2.54	−2.99	−0.08
0.7	−0.86	0.76	−0.02	0.46	−1.03	−0.08
0.9	−2.44	2.95	0.05	−1.36	0.99	0.00
0.99	−4.95	8.24	−0.73	−3.92	5.47	0.33
0.999	−7.32	15.41	−18.18	−6.29	11.76	0.92

Table 2: Pair corrections to μ^- decay spectrum *versus* x for $c = \pm 1$ and $P_\mu = 1$.

Let us define the relative contributions of pair corrections as

$$\begin{aligned}
\delta_2^{(0,e\bar{e})} &= \frac{L^2}{6}\left(\frac{\alpha}{2\pi}\right)^2 \frac{2f_2^{(0,\mathrm{NS})}(x) + 3f_2^{(0,\mathrm{S})}(x) \pm cP_\mu(2g_2^{(0,\mathrm{NS})}(x) + 3g_2^{(0,\mathrm{S})}(x))}{f_{\mathrm{Born}}(x) \pm cP_\mu g_{\mathrm{Born}}}\,, \\
\delta_2^{(1,e\bar{e})} &= L\left(\frac{\alpha}{2\pi}\right)^2 \sum_{i=\mathrm{NS,S,int}} \frac{f_2^{(1,i)}(x) \pm cP_\mu g_2^{(1,i)}(x)}{f_{\mathrm{Born}}(x) \pm cP_\mu g_{\mathrm{Born}}(x)}\,, \\
\delta_3^{(0,e\bar{e})} &= \frac{L^3}{6}\left(\frac{\alpha}{2\pi}\right)^3 \sum_{j=\mathrm{NS,S}} \frac{f_3^{(0,j)}(x) \pm cP_\mu g_3^{(0,j)}(x)}{f_{\mathrm{Born}}(x) \pm cP_\mu g_{\mathrm{Born}}(x)}\,.
\end{aligned} \tag{5.27}$$

For two values of c, they are given in table 2. One can see that the next-to-leading pair corrections have the same order of magnitude as the leading ones. This feature has been observed earlier in pair corrections to other processes [35]. The functions, which describe the LL pair effect, have numerically small coefficients and are less divergent (at $x \to 1$ and $x \to 0$) than the NLL ones. This means that the calculations of the non-logarithmic terms in the second order pair corrections is desirable, although the pair corrections are typically less than the photonic ones (at the same order in α).

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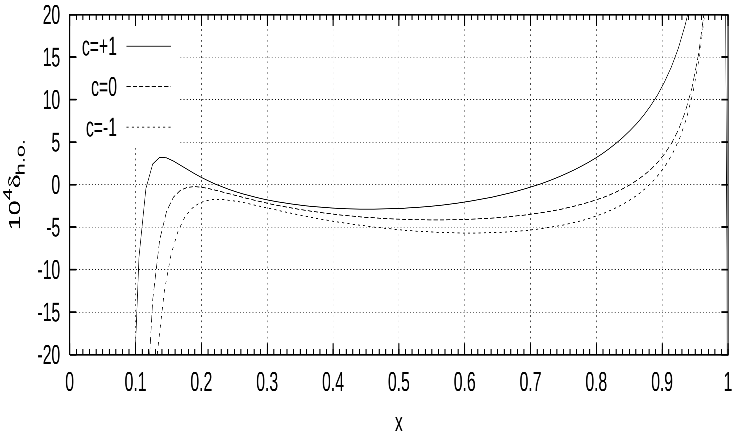

Figure 1: The relative effect of higher order corrections *versus* electron energy fraction for different angles.

The combined effect of the discussed above higher order corrections to μ^- decay spectrum is shown in figure 1:

$$\delta_{\text{h.o.}} = \delta_2^{(0,\gamma)} + \delta_2^{(1,\gamma)} + \delta_3^{(0,\gamma)} + \delta_2^{(0,e\bar{e})} + \delta_2^{(1,e\bar{e})} + \delta_3^{(0,e\bar{e})} + \delta_{\text{SF}}^{\text{exp}} \,. \tag{5.28}$$

One can see from the figure that the typical effect is of the order of $5 \cdot 10^{-4}$ for the intermediate range of x, and the effect becomes even larger at the two ends of the energy spectrum. So the corrections under consideration are really important for the modern and future experimental studies of the muon decay spectrum, where elaborated technique and high statistics allow to reduce the experimental errors to the level of $1 \cdot 10^{-4}$ and better.

6. Conclusions

To estimate the theoretical uncertainty in the description of the polarized muon decay spectrum by eqs. (2.1) and (5.1), we should consider the contributions, which have been omitted in the present calculations. They are: the $\mathcal{O}\left(\alpha^2\right)$ order terms, which are not enhanced by the large logarithm; sub-leading contributions in the third order $\mathcal{O}\left(\alpha^3 L^m\right)$, $m \leq 2$; and all the leading and sub-leading effects in the forth and higher orders ($\mathcal{O}\left(\alpha^n L^m\right)$ where $n \geq 4$, $m \leq n$) except those ones, which are taken into account by exponentiation. The possible contribution to the uncertainty from strong interactions is negligible in our case, since it is suppressed at least by $(\alpha/\pi)^2$, and the lowest order the contribution of hadronic virtual pairs was found in ref. [36] to be small itself.

An estimate of the omitted contributions by a simple counting of powers of the fine-structure constant and the large logarithm is not very safe, because there could be some extra enhancement factors, like numerically large constant coefficients or powers of $\ln(1-x)$ (the latter is partially taken into account by exponentiation). I suggest to estimate the omitted terms by a linear extrapolation of the known expansions in α and L. Namely,

$$\delta_2^{(2)} \sim \delta_2^{(1)} \frac{\delta_2^{(1)}}{\delta_2^{(0)}} \,, \qquad \delta_3^{(1)} \sim \delta_2^{(1)} \frac{\delta_3^{(0)}}{\delta_2^{(0)}} \,, \qquad \delta_4^{(0)} \sim \delta_3^{(0)} \frac{\delta_3^{(0)}}{\delta_2^{(0)}} \,, \tag{6.1}$$

where $\delta_2^{(2)}$ denotes the contribution of the second order terms, which are not enhanced by any large logarithm; $\delta_3^{(1)}$ is the third order next-to-leading correction (which can be calculated using the fragmentation function method described above); $\delta_4^{(0)}$ stands for the fourth order LL effect. This estimate of the theoretical error, can be applied to any particular set of experimental conditions to derive the actual uncertainty. In principle, the latter can depend on various cuts and details of particle registration and event selection (see discussion in ref. [14]). The approach (6.1) to estimate the uncertainty in the muon decay spectrum description works well for the main part of the kinematical domain. But, if one studies separately the extreme region $x \ll 1$ (or $1 - x \ll 1$), a special investigation of the convergence properties of our perturbative expansions in α and L should be performed. Evaluation of the uncertainties for any concrete experiment can be done using the analytical results and applying specific conditions of particle registration and data fitting.

The new results of ref. [18] and the present paper reduce the theoretical uncertainty in the description of the polarized muon decay spectrum. For the *quasi-realistic* experimental setup described in ref. [14], we can obtain now about 1.5 times better precision, so that, for instance, the theoretical uncertainty for the Michel parameter ρ becomes $2 \cdot 10^{-4}$ instead of $3 \cdot 10^{-4}$ obtained in ref. [14]. Nevertheless, this is still worse than the experimental precision $1 \cdot 10^{-4}$ planned at $\mathcal{TWIST}$ [3, 4]. Assuming that a theoretical precision of about one third (or less) of the experimental one would not spoil results of an experiment, we see a challenge for further investigations. First of all, a calculation of the $\mathcal{O}\left(\alpha^2\right)$ contributions, which are not enhanced by the large logarithm, is required to ameliorate the theoretical precision. This calculation is difficult, but possible by means of the standard methods.

The formulae for higher order corrections (with simple substitutions) are valid also for the decays of τ-lepton: $\tau \to \mu\nu_\tau\bar{\nu}_\mu$ and $\tau \to e\nu_\tau\bar{\nu}_e$.

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